Entanglement in Quantum Mechanics

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0. Introduction

Entanglement is one of the unique features of Quantum Mechanics in comparison to classical physics. Many of the novel phenomena and applications that are found in Quantum Cryptography, Quantum Information, and Quantum Computing arise from the concept of entanglement (see [1]).

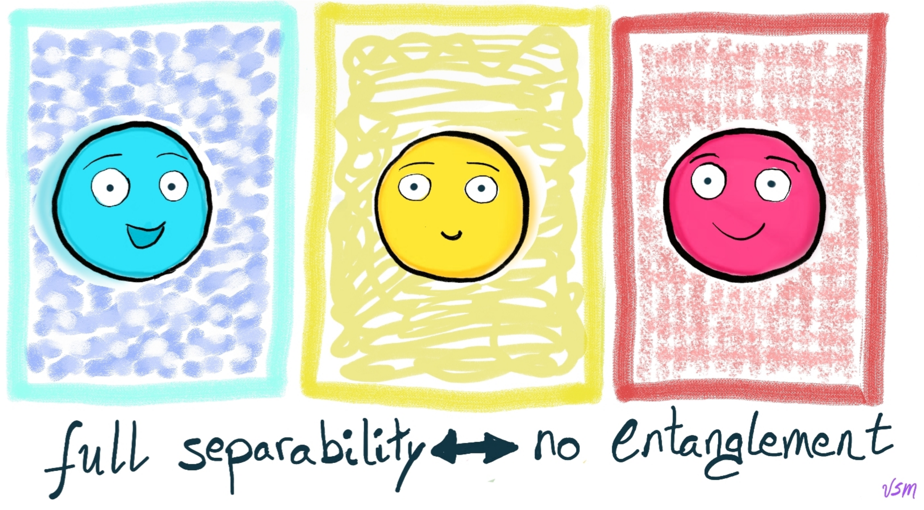
The present poster illustrates how entanglement is defined, and how it is usually quantified for 2-qubit and 3-qubit pure quantum states.

1. Definition of entanglement

Entanglement is defined for what it is not: **a quantum state is entangled if it cannot be decomposed as a tensor product for all parties (fully separable state)**. For a pure state $|\Psi\rangle$, it is entangled if:

$$|\Psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_N\rangle \quad (1)$$

The definition of entanglement is clear, but it does not provide a measure of *how much entanglement* there is in a given state. The quantification of entanglement in a multi-party scenario is not straightforward. However, for bi-partite and tri-partite states, the situation has been well-studied.



2. The concurrence C

One of the possible quantities to quantify the entanglement present in a 2-qubit state (bi-partite) is the **concurrence C**. For a pure state $|\psi\rangle$:

$$C(|\psi\rangle) = |\langle\psi|\, \sigma_y \otimes \sigma_y\, |\psi\rangle| \quad (2)$$

where σ_y is the Pauli matrix (also known as flip matrix)

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The concurrence takes values between 0 and 1: $1 \geq C \geq 0$.

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For example, if the 2-qubit state is $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ with $a, b, c, d \in \mathbb{R}$ and $a^2 + b^2 + c^2 + d^2 = 1$ the concurrence takes a very simple form:

$$C(|\psi\rangle) = 2|ad - bc| \quad (3)$$

- If $ad = bc \rightarrow c = \frac{ad}{b}$ $(b \neq 0)$, the state is fully separable (no entanglement) $|\psi\rangle = (b|0\rangle + d|1\rangle) \otimes (\frac{a}{b}|0\rangle + |1\rangle) \Rightarrow C = 0$
- If $|ad - bc| = \frac{1}{2}$, the state is maximally entangled, e.g. $|\psi\rangle = (|00\rangle + |11\rangle)\sqrt{2} \Rightarrow C = 1$

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C is related to the widely-known Von Neumann entropy S (or entropy of formation) as:

$$S = h\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}\right) \quad (4)$$

where h is the binary Shannon entropy function $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

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Generally, if the state is represented by a density matrix ρ_{AB}, $C(\rho_{AB})$ can still be computed but it has a more-complicated form than in (2) (see next section with the tangle τ).

3. The tangle τ

Related to the concurrence is the **tangle** τ. For a bi-partite state AB, it is defined as:

$$\tau_{AB} = max(\{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4},\ 0\})^2 \quad (5)$$

where λ_i are the eigenvalues (in descending order) of the matrix $\rho_{AB}\tilde{\rho}_{AB} = \rho_{AB}(\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$

The τ is just the square of the concurrence $\tau_{AB}(\rho_{AB}) = C(\rho_{AB})^2$. It is also normalised to 1. Its use is justified because it reveals an interesting feature when applied to a 3-qubit pure state $\rho_{ABC} = |\Psi\rangle\langle\Psi|$. The τ satisfies the next inequality (see [2]):

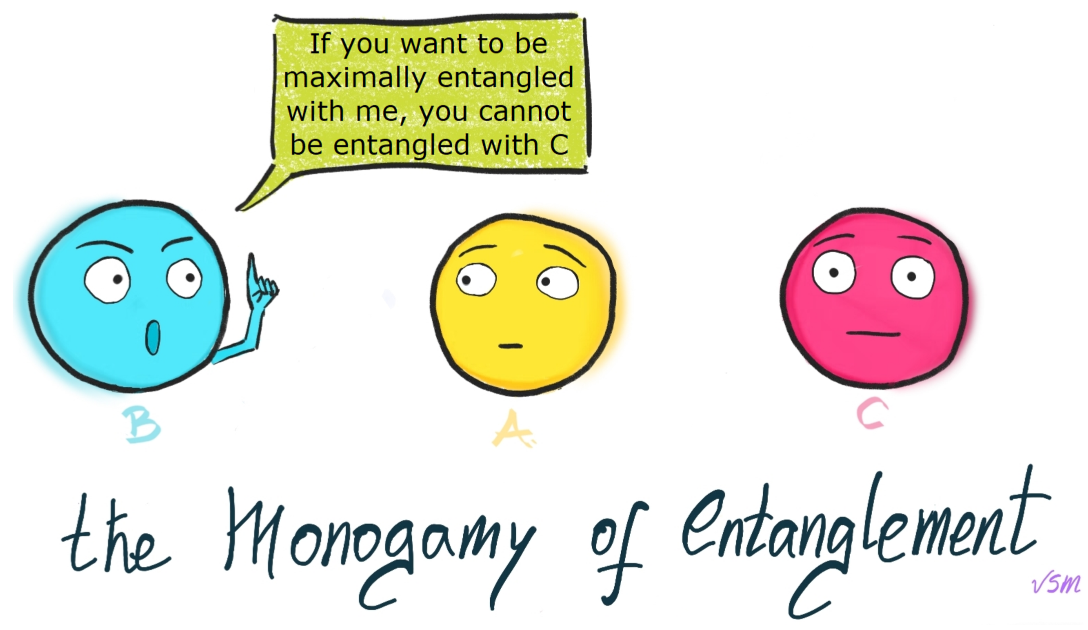
$$\tau_{AB} + \tau_{AC} \leq \tau_{A(BC)} \quad (6)$$

- τ_{AB} is the tangle of parties A and B. It is computed from $\rho_{AB} = Tr_C(\rho_{ABC})$. Similar for τ_{AC}.
- $\tau_{A(BC)}$ is the amount of entanglement of party A with party BC. For a pure state, $\tau_{A(BC)} = 4Det(\rho_A)$.

Inequality (6) has important physical implications:

Party A sharing some entanglement with BC constrains the amount of entanglement that A can share individually with B and with C.

Equivalently, the more entangled parties A and B are, the less entangled parties A and C can be.



This constraint is known as **the monogamy of entanglement**

The **3-tangle** τ_{ABC} is defined as the difference between the rhs and the lhs in (6):

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC} \quad (7)$$

The value of τ_{ABC} does not depend on the chosen party in the rhs of (7). In fact, τ_{ABC}, up to a constant, is the absolute value of the Cayley hyperdeterminant (invariant) of the 3-qubit state. Thus, τ_{ABC} is a good measure of the genuine 3-party entanglement present in a 3-qubit pure state.

3.0. Example of tangle computation

Figure 1 and 2 show the values of the τ for a GHZ-like state $|\Psi\rangle = \alpha|000\rangle + \sqrt{1-|\alpha|^2}\,|111\rangle$ and a W-like state $|\Psi\rangle = \beta(|001\rangle + |010\rangle) + \sqrt{1-2|\beta|^2}\,|100\rangle$ as a function of $|\alpha|$ and $|\beta|$, respectively.

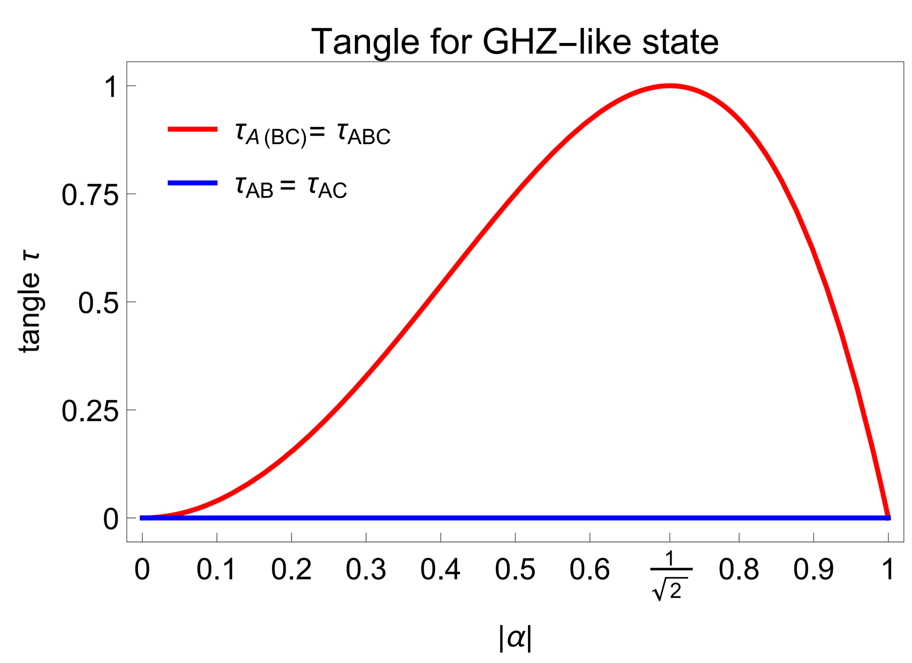


Fig. 1

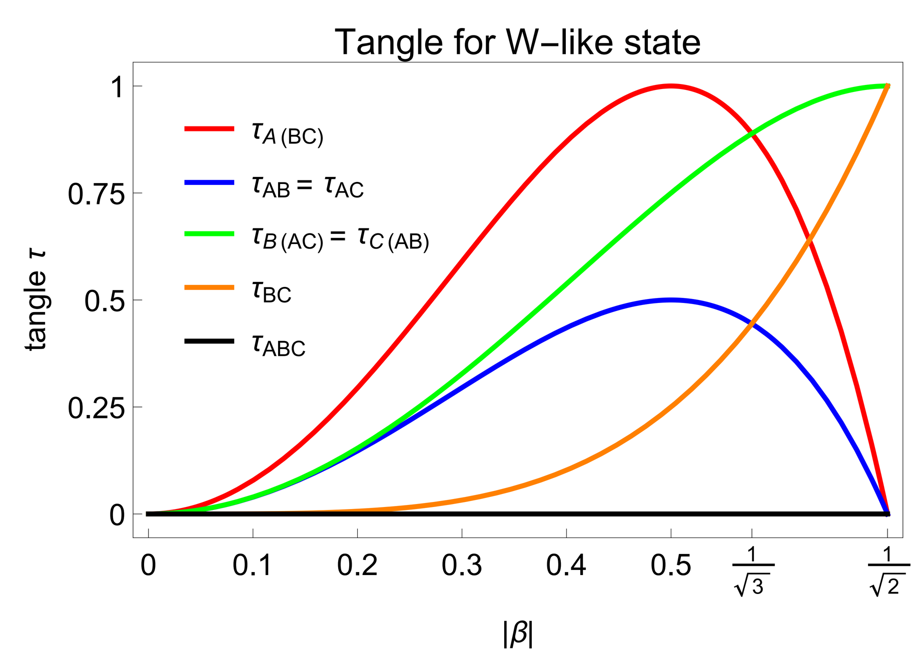


Fig. 2

References

[1] Ryszard Horodecki et al. "Quantum entanglement". In: *Reviews of Modern Physics* 81.2 (2009), pp. 865–942.

[2] Valerie Coffman, Joydip Kundu, and William K. Wootters. "Distributed entanglement". In: *Physical Review A* 61.5 (2000).